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Growth Models and Their Stability Analysis for Populations of a Single Species

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Population Models and Their Dynamic Analysis on Equilibriums and Stabilities

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Introduction to Population Models

Population models

Use typical features of population changes to set up mathematical models for analyzing population growth characteristics and making predictions of future populations.

Benefits of population models:

♥ Useful tools for understanding, explaining, and predicting the dynamics and persistence of biological populations.

♥ Used for assessing the status of a population, diagnosing causes of population declines or explosive growth, prescribing management targets, and evaluating the prognosis of a population's likely responses to alternative management actions.

Continuous Models

Continuous population models:

♣ **The population changes continuously** at every moment under consideration of main factors including birth, death, migration, resources, etc.

♣ **Differential equations with Calculus** are mainly employed to set up the models

Models in the talk:

♣**The exponential growth model**: considers a linear growth parameter to represent population change rate;

♣ **The logistic model**: considers a linear growth parameter and the carrying capacity of the environment to prevent population growth explosion;

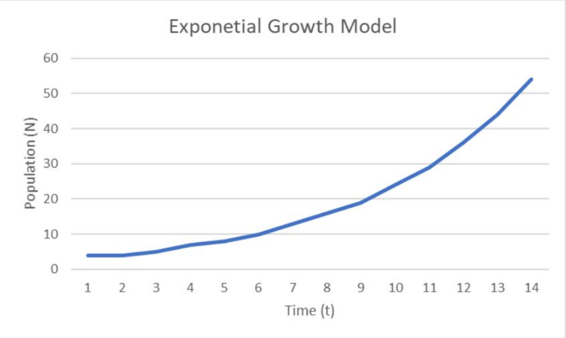
♣ **The delay model**: considers young people with a time delay to reach birth capability.

Exponential Model

- The model considers **annual growth rate** simply determined by a liner birth rate “b” and a death rate “d” without considering other factors.
- The annual growth rate is expressed by the derivative of the population function N(t) as dN/dt.
- A particular year is considered as an initial year, and the initial population is denoted by N_0.
- The model in the differential-equation form and the solution for the population N(t) in the exponential form are

$$\frac{dN}{dt} = bN - dN \xrightarrow{\text{yields}} N(t) = N_0 e^{(b-d)t}$$

- The population increases to infinity without upper boundary. This is unrealistic, because population growth is constrained also by resources and many other factors.



Logistic Model

- This model considers the carrying capacity K of the environment determined by available resources. It is a self-limiting process that should operate when a population becomes too large. The model is:

$$\frac{dN}{dt} = \mathrm{rN}\left(1 - \frac{\mathrm{N}}{\mathrm{K}}\right)$$

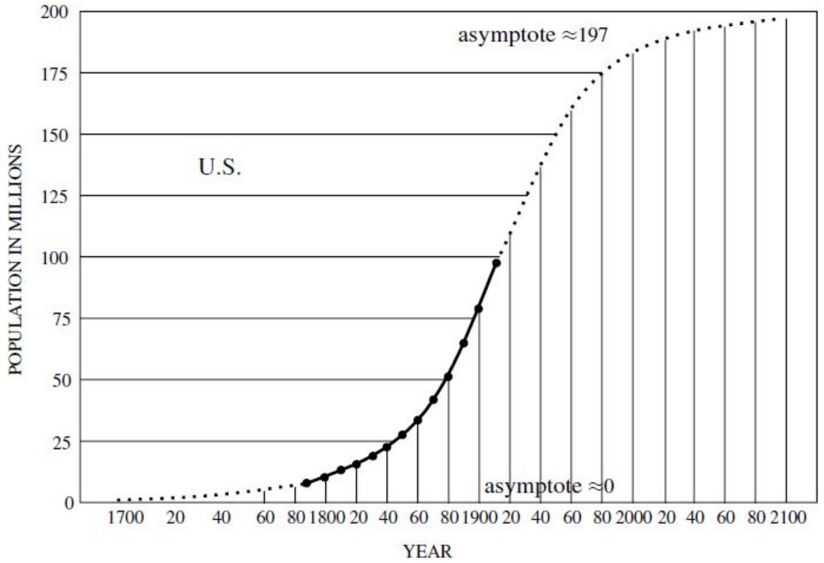
- Population Equilibriums: The constant solutions of the population differential equations. For the logistic model, the equilibriums are:

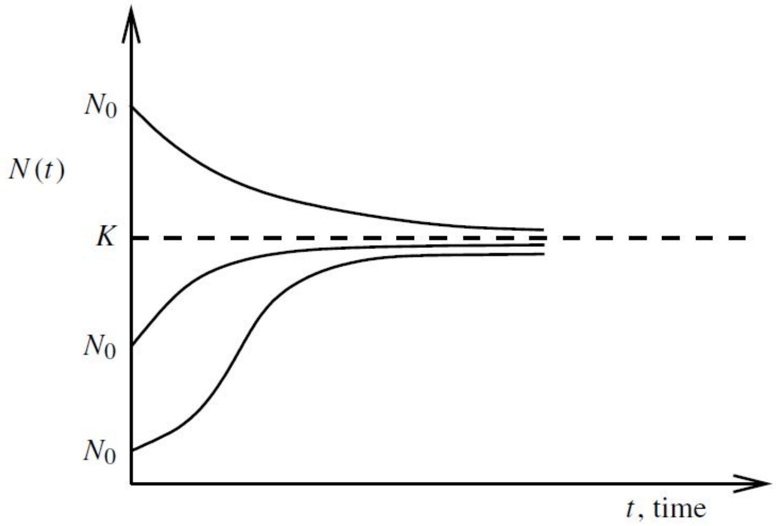
  N(t)=0, N(t)=K.

- Stable equilibrium: when the initial population is near the equilibrium, the population will eventually approach the equilibrium. (Otherwise, the equilibrium is unstable.)
- N(t)=0 is an unstable equilibrium; the other equilibrium N(t)=K is stable. For any non-zero initial population, the population increases/decreases until it reaches the constant K, the carrying capacity of the environment.
- The population function is $N(t) = \frac{N_0 K e^{rt}}{K+N_0(e^{rt}-1)} \Rightarrow K$ as $t \to \infty$

Logistic Model on US Population

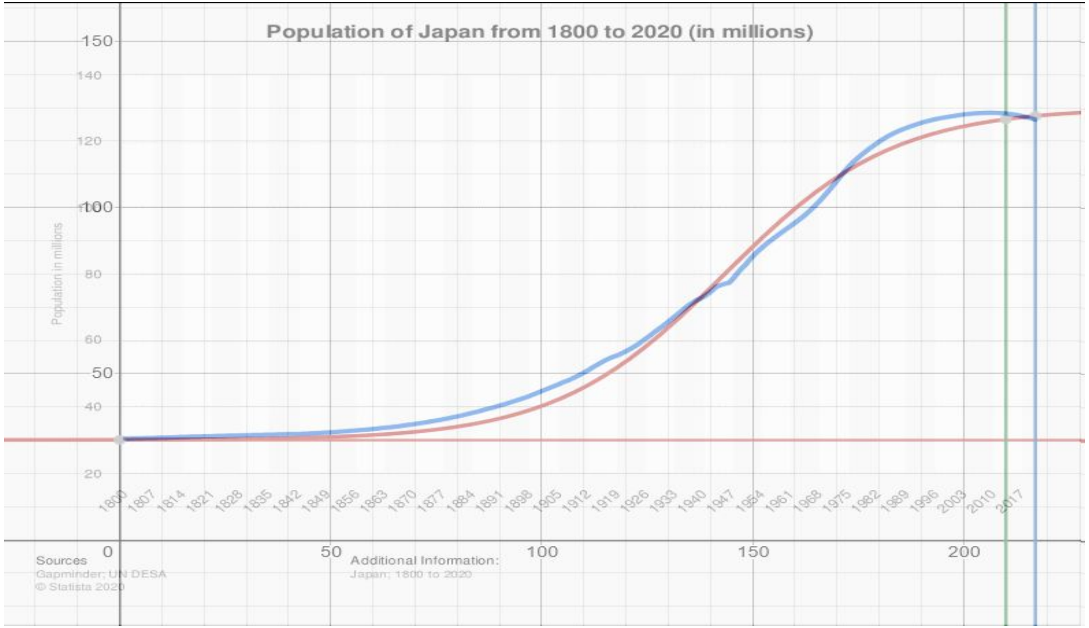
- The left panel: the population curve compared with the realistic US population in the 19th century with a good match. The carrying capacity is 197 million.
- The right panel: stability of the equilibrium N(t)=K.





Logistic Model on Japanese Population

- Population Study of Japan: 1800 - 2020.
- The true population of Japan is in blue; The Calculated Population of Japan is red



$$N(t) = \frac{N_0 K e^{rt}}{K + N_0 (e^{rt} - 1)} \Rightarrow K \text{ as } t \to \infty$$

Delay Model

- Updated from the logistic model, considering the young people need a delay time T to give birth.
- The present growth rate at time t is partially influenced by the population at the past time t-T. The model is below

$$\frac{dN}{dt} = rN(t)\left[1 - \frac{N(t-T)}{K}\right]$$

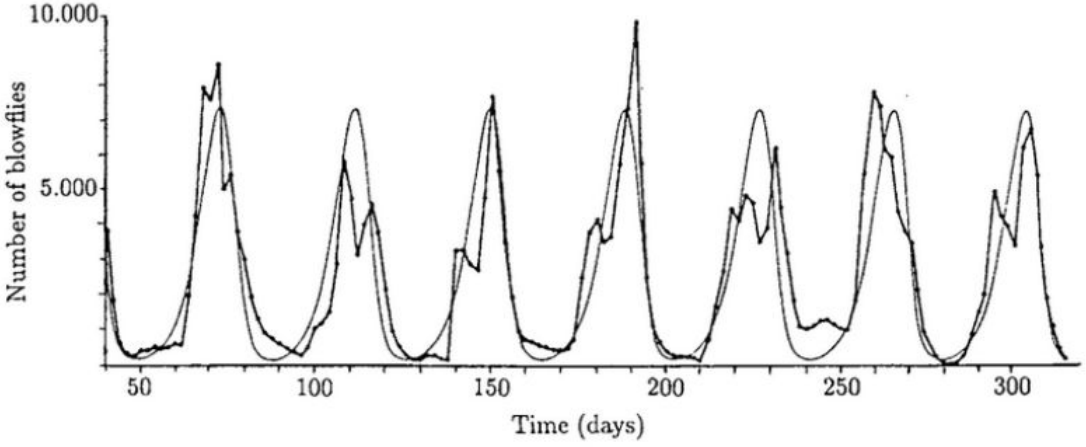
- The equilibriums are the same as in the logistic model. However, none of the equilibriums are stable. The solution N(t) is as below:

$$N(t) = 1 + \mathrm{Re}\{c\exp[\delta t + i(1+\sigma)t]\}$$
$$\approx 1 + \mathrm{Re}\left\{c\exp\left[\frac{\varepsilon t}{1+\frac{\pi^2}{4}}\right]\exp\left[it\left\{1 - \frac{\varepsilon\pi}{2(1+\frac{\pi^2}{4})}\right\}\right]\right\}$$

"Re" means the real part of the complex number with "i" being the complex unit. An illustrative curve is shown in Fig.3. The population exhibits a periodic behavior due to the delay effect [4].

Delay Model Population Curve

- The population curve with delay time shows a periodic pattern.
- Blowflies population: The smooth curve is for the theoretical results, and the curve with many turning points are for the real population data.



Summary and Discussion

♣ All three models describe population growth using the main characteristics of growth; patterns of the population quantity in periods matching real population data.

♣ The models provide dynamic analysis for long-term population behaviors: equilibriums, stability, and population progression from small perturbations.

♣ The parameters can be time-dependent instead of being stationary. For example, the birth/death rate can change annually; other factors include: migration, emigration, pandemic, etc.

♣ Population models can also be used to describe bacteria, virus, and infection growth.

♣ Population models are helpful for making public policy and assigning resources when knowing the growth pattern in the future.

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Thank you for your attention